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AFDELING NUMERIEKE WISKUNDE NW 126/82 MEI
(DEPARTMENT OF NUMERICAL MATHEMATICS)

M. BAKKER

ONE-DIMENSIONAL GALERKIN METHODS AND SUPERCONVERGENCE AT INTERIOR NODAL POINTS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65N30

CR Category: 5.17

One-dimensional Galerkin methods and superconvergence at interior nodal points *)

by

M. Bakker

ABSTRACT

In the case of one-dimensional Galerkin methods the phenomenon of superconvergence at the knots is already known for years [5,7]. In this paper, a minor kind of superconvergence at specific points inside the segments of the partition is discussed for two classes of Galerkin methods: the Ritz-Galerkin method for 2m-th order self-adjoint boundary problems and the collocation method for arbitrary m-th order boundary problems. These interior points are the zeros of the Jacobi polynomial $P_n^{m,m}(\sigma)$ shifted to the segments of the partition; $n = k+1-2m$, where k is the degree of the finite element space. The order of convergence at these points is k+2, one order better than the optimal order of convergence. Also, it can be proved that the derivative of the finite element solution is superconvergent of $O(h^{k+1})$ at the zeros of the Jacobi polynomial $P_{n+1}^{m-1,m-1}(\sigma)$ shifted to the segments of the partition. This is one order better than the optimal order of convergence for the derivative.

KEY WORDS & PHRASES: *Galerkin methods, collocation methods, finite element method, superconvergence, Jacobi polynomials*

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*) This report will be submitted for publication elsewhere,

1. INTRODUCTION

We consider the two-point boundary problem

$$(1.1)\qquad \begin{aligned} &-(p(x)y')' + q(x)y = f(x), \qquad x \in [-1,+1] = I; \\ &y(\pm 1) = 0, \end{aligned}$$

where $p(x) > 0$, $q(x) \geq 0$ and $f(x)$ are sufficiently smooth. Let

$$(1.2)\qquad \begin{aligned} &\Delta = \{-1 = x_0 < x_1 < \ldots < x_N = 1\}; \\ &x_j = -1 + hj; \quad j = 0,\ldots,N; \quad h = 2/N; \\ &I_j = [x_{j-1}, x_j], \quad j = 1,\ldots,N \end{aligned}$$

be a uniform partion of I and define $M_0^{k,0}(\Delta)$ by

$$(1.3)\qquad M_0^{k,0}(\Delta) = \{V \mid V \in C^0(I);\ V \in P_k(I_j),\ j = 1,\ldots,N;\ V(\pm 1) = 0\}$$

where for any interval E, $P_k(E)$ denotes the space of polynomials of degree k restricted to E. Then the finite element approximation $Y \in M_0^{k,0}(\Delta)$ of y is determined by

$$(1.4)\qquad (pY',V') + (qY,V) = (f,V), \quad V \in M_0^{k,0}(\Delta),$$

where (,) denotes the $L^2(I)$ inner product. It has the following convergence properties [7]

$$(1.5)\qquad \begin{aligned} &\|y-Y\|_\ell \leq C_1 h^{k+1-\ell}\|y\|_{k+1}, \qquad \ell = 0,1; \\ &|(y-Y)(x_j)| \leq C_2 h^{2k}\|y\|_{k+1}, \qquad j = 1,\ldots,N-1, \end{aligned}$$

where C_1 and C_2 are positive constants and where

$$(1.6)\qquad \begin{aligned} &\|v\|_\ell = [\sum_{j=0}^{\ell} (D^j v, D^j v)]^{\frac{1}{2}}, \qquad \ell \geq 0; \\ &D^j v = \frac{d^j v}{dx^j}, \qquad j \geq 0. \end{aligned}$$

Also, it is known [3] that for specific points inside I_j, Y has the error bound

$$(1.7)\qquad \begin{aligned} &|(y-Y)(\xi_{j\ell})| \le C(y)h^{k+2} \\ &\xi_{j\ell} = x_{j-1} + \frac{h}{2}(1+\sigma_\ell), \quad \ell = 1,\dots,k-1;\ j = 1,\dots,N, \end{aligned}$$

where $\sigma_1,\dots,\sigma_{k-1}$ are the zeros of $P_k'(\sigma)$, $P_k(\sigma)$ the k-th degree Legendre polynomial. This is one order better than the optimal error bound which is of $O(h^{k+1})$.

It is this phenomenon of so-called *interior superconvergence* on which we will concentrate our attention. In the next two sections, we will treat two classes of finite element methods where this occurs: the Ritz-Galerkin and the collocation method [8]. Also, we will use that superconvergence to give a new proof of the superconvergence of the derivative at other Gaussian points [6].

Before that, we give some definitions we need throughout this paper.

For any $E \subset I$ and $m \ge 0$, we define

$$(1.8)\qquad \begin{aligned} &\|v\|_{H^m(E)} = \Big[\sum_{\ell=0}^{m} (D^\ell v, D^\ell v)_{L^2(E)}\Big]^{\frac{1}{2}}; \\ &\|v\|_{W^m(E)} = \sum_{\ell=0}^{m} \|D^\ell v\|_{L^\infty(E)}; \\ &W^m(E) = \{v \mid D^\ell v \in L^\infty(E),\ \ell = 0,\dots,m\}; \\ &H^m(E) = \{v \mid D^\ell v \in L^2(E),\ \ell = 0,\dots,m\}. \end{aligned}$$

Also, we define the Δ-related norms

$$(1.9)\qquad \begin{aligned} &\|v\|_{m,\Delta} = \Big[\sum_{j=1}^{N}\sum_{\ell=0}^{m} (D^\ell v, D^\ell v)_{L^2(I_j)}\Big]^{\frac{1}{2}}; \\ &\|v\|_{W^m(\Delta)} = \max_{j=1,\dots,N} \|v\|_{W^m(I_j)}. \end{aligned}$$

Finally, throughout this paper, C, C_1, etc. will be positive constants, not the same at each occurrence.

2. THE RITZ-GALERKIN METHOD

Consider the 2m-th order two-point boundary problem

$$\text{(2.1)}\qquad \begin{aligned} &Lu \equiv \sum_{\ell=0}^{m} (-1)^{\ell} D^{\ell}[p_{\ell}(x)D^{\ell}u] = f(x), \quad x \in I; \\ &D^{\ell}u(\pm 1) = 0, \quad \ell = 0,\dots,m-1, \end{aligned}$$

where $p_0,\dots,p_m$ and f are sufficiently smooth in x. We assume that there exists some $C > 0$ with the property

$$\text{(2.2)}\qquad \begin{aligned} &B(v,v) \geq C\|v\|_m^2; \quad v \in H_0^m(I); \\ &B(u,v) = \sum_{\ell=0}^{m} (p_{\ell}D^{\ell}u, D^{\ell}v); \quad u,v \in H_0^m(I); \\ &H_0^m(I) = \{v \mid v \in H^m(I);\ D^{\ell}v(\pm 1) = 0,\ \ell = 0,\dots,m-1\}, \end{aligned}$$

in other words, $B(,)$ is strongly coercive.

For some partition Δ of I defined by (1.2) and some integer $k \geq 2m-1$, we define the finite element space

$$\text{(2.3)}\qquad M_0^{k,m}(\Delta) = \{V \mid V \in H_0^m(I);\ V \in P_k(I_j),\ j = 1,\dots,N\}$$

The solution u of (2.1) can be approximated in $M_0^{k,m}(\Delta)$ by the solution U of the weak Galerkin form

$$\text{(2.4)}\qquad B(U,V) = (f,V), \quad V \in M_0^{k,m}(\Delta).$$

The error function $e = u - U$ has the bounds [2,4]

$$\text{(2.5)}\qquad \begin{aligned} &\|e\|_{\ell} \leq Ch^{k+1-\ell}\|u\|_{k+1}, \quad \ell = 0,\dots,m; \\ &|D^{\ell}e(x_j)| \leq Ch^{2r}\|u\|_{k+1}, \quad \ell = 0,\dots,m-1;\ j = 1,\dots,N-1; \\ &r = k+1-m. \end{aligned}$$

What we want to prove is the fact that *inside* each segment I_j there exist $n = k+1 - 2m$ distinct and specific points where $|e(x)|$ is of $O(h^{k+2})$, one order better than the optimal order of convergence. This is, of course, only true, if $n \geq 1$ or $k \geq 2m$. These points are shown to be the zeros of the Jacobi polynomial $P_n^{m,m}(\sigma)$, which will be introduced in the next section.

2.1. The Jacobi polynomial

The Jacobi polynomial $P_n^{\alpha,\beta}(\sigma)$ is defined by Rodrigues' formula [1] as

$$(2.6)\qquad \begin{aligned} P_n^{\alpha,\beta}(\sigma) &= [w(\sigma)]^{-1}D^n[(1-\sigma^2)^n w(\sigma)]A_n^{\alpha,\beta}, \qquad n \geq 0; \\ w(\sigma) &= (1-\sigma)^\alpha(1+\sigma)^\beta; \qquad \alpha,\beta > -1; \end{aligned}$$

where $A_n^{\alpha,\beta}$ is some normalizing factor, e.g. such that $P_n^{\alpha,\beta}(1) = 1$ or $P_n^{\alpha,\beta}(1) = (1+\alpha)(1+\frac{\alpha}{2}) \ldots (1+\frac{\alpha}{n})$. It has the important property

$$(2.7)\qquad (wP_i^{\alpha,\beta}, P_j^{\alpha,\beta}) = \delta_{ij}(wP_i^{\alpha,\beta}, P_i^{\alpha,\beta}), \qquad 0 \leq i,j,$$

where δ_{ij} is the Kronecker symbol.

From now on, we are only interested in the case that $\alpha = \beta = m$, where m is some nonnegative integer. In that case, we replace the double superscript m,m by the single superscript m.

LEMMA 1. *Let the linear interpolation* $\Pi: C^{m-1}(I) \to P_{n+2m-1}(I)$ *be determined by*

$$(2.8)\qquad \begin{aligned} D^\ell(\Pi f)(\pm 1) &= D^\ell f(\pm 1), \qquad \ell = 0,\ldots,m-1 \\ (\Pi f)(\sigma_{in}^m) &= f(\sigma_{in}^m), \qquad i = 1,\ldots,n \end{aligned}$$

and let the integral $\int_{-1}^{+1} f(\sigma)d\sigma$ be approximated by the quadrature formula

$$(2.9)\qquad \int_{-1}^{1} f(\sigma)d\sigma \doteq \int_{-1}^{1} (\Pi f)(\sigma)d\sigma$$

Then (2.9) is exact if $f \in P_{2r-1}(I)$, with $r = m+n$.

PROOF. From (2.8), it follows that there exists a function $g(\sigma)$ such that

$$(2.10)\qquad f(\sigma) - (\Pi f)(\sigma) = (1-\sigma^2)^m P_n^m(\sigma) g(\sigma).$$

From (2.7), we know that

$$(2.11)\qquad ((1-\sigma^2)^m P_n^m, g) = 0, \quad \text{if } g \in P_{n-1}(I),$$

which completes the proof. □

Elaboration of (2.10) gives the formula

$$(2.12)\qquad \int_{-1}^{+1} (\Pi f)(\sigma)d\sigma = \sum_{\ell=0}^{m-1} [\theta_{\ell 1} D^{\ell} f(-1) + \theta_{\ell 2} D^{\ell} f(+1)] + \sum_{\ell=1}^{n} \omega_{\ell} f(\sigma_{\ell n}^m),$$

with

$$\omega_i = \int_{-1}^{+1} \Phi_i(\sigma)d\sigma; \qquad \Phi_i(\sigma) = \frac{(1-\sigma^2)^m \, P_n^m(\sigma)}{(\sigma-\sigma_{in}^m)[(1-\sigma^2)^m \frac{d}{d\sigma} P_n^m(\sigma)]_{\sigma=\sigma_{in}^m}};$$

$$(2.13)\qquad \theta_{\ell i} = \int_{-1}^{+1} \Psi_{\ell i}(\sigma)d\sigma; \qquad \Psi_{\ell i} \in P_k(I);$$

$$\Psi_{\ell i}(\sigma_{jn}^m) = 0; \qquad \ell = 0,\dots,m-1; \; i = 1,2;$$

$$D^s \Psi_{\ell i}((-1)^j) = \delta_{ij}\delta_{\ell s}; \qquad 1 \le i,j \le 2; \quad 0 \le \ell,s \le m-1.$$

The approximation error of (2.9) is $R_{mn} D^{2r} f(\xi)$, where R_{mn} depends on m and n only and where ξ lies inside I.

In the next section, we will use (2.9) - (2.12) to establish superconvergence of $O(h^{k+2})$ at the Jacobi points.

2.2. Superconvergence at Jacobi points

We return to problem (2.1) and its Ritz-Galerkin solution (2.4). It is standard that

$$(2.14) \qquad B(e,V) = 0, \quad V \in M_0^{k,m}(\Delta).$$

For $k \geq 2m$, we define for any I_j the n-dimensional subspace $S_0(I_j)$ of $M_0^{k,m}(\Delta)$ by

$$(2.15) \qquad S_0(I_j) = \{V \mid V \in H_0^m(I) \cap P_k(I_j);\ \mathrm{supp}(V) = I_j\}.$$

For $S_0(I_j)$, a basis can be constructed, consisting of the Lagrangian polynomials $\phi_i(x)$ defined by

$$(2.16) \qquad \phi_i(x) = \Phi_i(1 + 2(x-x_j)/h), \quad i = 1,\dots,n,$$

where Φ_i is defined by (2.13).

If we apply (2.14) to ϕ_i, we obtain after partial integration

$$(2.17) \qquad (e,L\phi_i) = \sum_{\ell=1}^{m} \sum_{\nu=0}^{\ell-1} [(-1)^{\nu+1} D^{\ell-\nu-1} e(x) D^{\nu}(p_\ell(x) D^\ell \phi_i(x))]_{x_{j-1}}^{x_j}, \qquad i = 1,\dots,n.$$

We now define the interior nodal points $\xi_{j\ell}$ by

$$(2.18) \qquad \xi_{j\ell} = x_{j-1} + \frac{h}{2}(1+\sigma_{\ell n}^m), \ \ell = 1,\dots,n,$$

where $\sigma_{\ell n}^m$ is the ℓ-th zero of $P_n^m(\sigma)$, as defined in §2.1.

Application of (2.12) to (2.17) gives

$$(2.19) \qquad \frac{h}{2} \sum_{\ell=1}^{n} \omega_\ell e(\xi_{j\ell}) L\phi_i(\xi_{j\ell}) = (e,L\phi_i) - \sum_{\ell=0}^{m-1} [\theta_{\ell 1} D^\ell e(x_{j-1}) + \theta_{\ell 2} D^\ell e(x_j)] (\tfrac{h}{2})^{\ell+1} + R_{mn} [D^{2r}(eL\phi_i)]_{x=\xi\in I_j} h^{2r+1}, \qquad i = 1,\dots,n.$$

where R_{mn} depends on m and n only. If we multiply both sides of (2.19) by $2h^{2m-1}$ and apply formula (2.5), we have

$$\left|\sum_{\ell=1}^{n}[\omega_\ell L\phi_i(\xi_{j\ell})h^{2m}]e(\xi_{j\ell})\right| \le C_1\sum_{\ell=0}^{m-1}(|D^\ell e(x_{j-1})| + |D^\ell e(x_j)|)$$

$$(2.20)\qquad + C_2h^{2m}\sum_{\ell=0}^{m-1}h^\ell(|D^\ell e(x_{j-1})| + |D^\ell e(x_j)|) + C_3h^{2k+2}\|eL\phi_i\|_{W^{2r}(I_j)}$$

$$\le C_1h^{2r}\|u\|_{k+1} + C_2(u)h^{k+2} \le C(u)h^{k+2}.$$

On the other hand, if we apply quadrature rule (2.9) to the inner product

$$(2.21)\qquad 2h^{2m-1}(\phi_\ell, L\phi_i) = 2h^{2m-1}B(\phi_i,\phi_\ell),$$

we find that

$$|2h^{2m-1}B(\phi_i,\phi_\ell) - h^{2m}\omega_\ell L\phi_i(\xi_{j\ell})|$$

$$(2.22)\qquad \le Ch^{2k+2}\|\phi_\ell L\phi_i\|_{W^{2r}(I_j)} \le Ch^2,$$

which means that $(h^{2m}\omega_\ell L\phi_i(\xi_{j\ell}))$ is an $O(h^2)$ perturbation of a positive definite matrix whose entries are of $O(1)$. From this, it easily follows that the entries of $(\omega_\ell L\phi_i(\xi_{j\ell})h^{2m})^{-1}$ are of $O(1)$. This completes the proof of

THEOREM 1. *Let* $u \in H_0^m(I) \cap H^{k+1}(I) \cap W^{2r}(\Delta)$ *be the solution of* (2.1) *and let* $U \in M_0^{k,m}(\Delta)$ *be the solution of* (2.4). *Then* $e = u - U$ *has the bounds* (2.5) *and the additional bound*

$$(2.23)\qquad |e(\xi_{j\ell})| \le C(u)h^{k+2}, \quad j = 1,\ldots,N;\ \ell = 1,\ldots,n,$$

where $\xi_{j\ell}$ *is defined by* (2.18) □

We can use the local convergence properties (2.5) and (2.23) to establish superconvergence properties of De at interior points of I_j. To that end, we define the projection Π_Δ: $H_0^m(I)^k \cap H^{k+1}(I) \to M_0^{k,m}(\Delta)$ by

$$(2.24)\qquad \begin{aligned} (\Pi_\Delta u)(\xi_{j\ell}) &= u(\xi_{j\ell}); \quad j = 1,\ldots,N;\ \ell = 1,\ldots,n; \\ D^\ell(\Pi_\Delta u)(x_j) &= D^\ell u(x_j), \quad j = 1,\ldots,N-1;\ \ell = 0,\ldots,m-1. \end{aligned}$$

Then on any I_j, $u - \Pi_\Delta u$ has the representation

$$(2.25)\qquad \begin{aligned} &u(x) - (\Pi_\Delta u)(x) = h^{k+1}(1-\sigma^2)^m P_n^m(\sigma) E_j(x); \\ &\sigma = \tfrac{2}{h}(x-\bar{x}_j); \qquad \bar{x}_j = \tfrac{1}{2}(x_{j-1} + x_j); \end{aligned}$$

where $E_j(x)$ and $E_j'(x)$ have bounds depending on j only. This property can be proved by expanding u and $\Pi_\Delta u$ as Taylor series around $\bar{x}_j$.

Differentiating (2.25), we obtain

$$(2.26)\qquad D(u - \Pi_\Delta u)(x) = h^{k+1}E_j'(x)(1-\sigma^2)^m P_n^m(\sigma) + 2h^k E_j(x)\frac{d}{d\sigma}(1-\sigma^2)^m P_n^m(\sigma) .$$

From [1], we know that

$$(2.27)\qquad \begin{aligned} &P_n^m(\sigma) = A_{mn}\frac{d}{d\sigma}P_{n+1}^{m-1}(\sigma); \\ &\frac{d}{d\sigma}[(1-\sigma^2)^m \frac{d}{d\sigma}P_{n+1}^{m-1}(\sigma)] = B_{mn}(1-\sigma^2)^{m-1}P_{n+1}^{m-1}(\sigma), \end{aligned}$$

where A_{mn} and B_{mn} depend on m and n only. From (2.26) and (2.27) we can conclude that

$$(2.28)\qquad \begin{aligned} &|D(u - \Pi_\Delta u)(x)| = O(h^{k+1}), \qquad \text{if } x = \eta_{j\ell}; \\ &\eta_{j\ell} = x_{j-1} + \tfrac{h}{2}(1 + \sigma_{\ell n+1}^{m-1}), \qquad j = 1,\dots,N;\ \ell = 1,\dots,n+1 \end{aligned}$$

Consider now $U - \Pi_\Delta u$. From (2.5), (2.23) and (2.24), we can conclude that

$$(2.29)\qquad \|U - \Pi_\Delta u\|_{L^\infty(I)} \le C(u)h^{k+2}.$$

From (2.26) - (2.27), one easily proves

<u>THEOREM 2</u>. *Let the conditions of Theorem* 1 *holds. Then* e(x) *has the additional bound*

$$(2.30) \qquad |De(\eta_{j\ell})| \le C(u)h^{k+1},$$

where $\eta_{j\ell}$ *is defined by* (2.28). *This is one order better than the optimal order of convergence for* $e'(x)$. □

2.3. Quadrature rules

Without giving proofs, we state that all the local convergence properties from the Theorems 1 and 2 are preserved whenever (,) is replaced by some approximating quadrature $(,)_h$ which is of $O(h^q)$, $q \ge 2r$, i.e.

$$|(\alpha,\beta) - (\alpha,\beta)_h| \le C(\alpha,\beta)h^q, \qquad q \ge 2r.$$

Examples are the extended r-point Gauss-Legendre rule or the extended (r+1)-point Lobatto rule.

3. COLLOCATION METHODS

We consider the m-th order boundary problem

$$(3.1) \qquad \begin{aligned} &Lu(x) \equiv D^m u(x) + \sum_{i=0}^{m-1} p_i(x) D^i u(x) = f(x), \qquad x \in I; \\ &\beta_\ell[u] = 0, \quad \ell = 1,\dots,m, \end{aligned}$$

where $p_0,\dots,p_{m-1}$ and f are sufficiently smooth functions and where $\beta_1,\dots$ $\dots,\beta_m$ are continuous linear functionals over $C^{m-1}(I)$. We note that the functions $p_0,\dots,p_{m-1}$ and f and the operator L are not the same as in the previous chapter. We assume that (3.1) has a unique solution and that $\beta_1,\dots$ $\dots,\beta_m$ are linearly independent over $P_{m-1}(I) = \ker(D^m)$.

Let Δ be a partition of I defined by (1.2). Then, for $k \ge 2m-1$, we define the finite element space $S_0^{k,m}(\Delta)$ by

$$(3.2) \qquad \begin{aligned} &S_0^{k,m}(\Delta) = \{V \mid V \in C_0^{m-1}(I);\ V \in P_k(I_j),\ j = 1,\dots,N\}; \\ &C_0^{m-1}(I) = \{v \mid v \in C^{m-1}(I);\ \beta_\ell[v] = 0,\ \ell = 1,\dots,m\}. \end{aligned}$$

The collocation solution $U \in S_0^{k,m}(\Delta)$ of (3.1) is defined as follows.

For $r = k+1-m$, we define the collocation points $z_{j\ell}$ by

$$z_{j\ell} = x_{j-1} + \frac{h}{2}(1+\sigma^0_{\ell r}), \quad j = 1,\ldots,N; \ \ell = 1,\ldots,r, \tag{3.3}$$

where $\{\sigma^0_{\ell r}\}$ are the zeros of the r-th degree Legendre polynomial $P^0_r(\sigma)$. Then U is determined by the linear system

$$LU(z_{j\ell}) = f(z_{j\ell}), \qquad j = 1,\ldots,N; \ \ell = 1,\ldots,r. \tag{3.4}$$

The error function $e = u - U$ has the bounds [5]

$$\begin{aligned} &\|e\|_\ell \le Ch^{k+1-\ell}\|u\|_{k+1}, \qquad \ell = 0,\ldots,m; \\ &|D^\ell e(x_j)| \le C(u)h^{2r}, \qquad \ell = 0,\ldots,m-1; \ j = 0,\ldots,N. \end{aligned} \tag{3.5}$$

In order to establish superconvergence at interior points of I_j [8], we recall the n-dimensional subspace $S_0(I_j)$ of $S_0^{k,m}(\Delta)$ defined by (2.15). For any $V \in S_0(I_j)$, we have, if we put $p_m(x) \equiv 1$,

$$(e,L^TLV) = (Le,LV) + \sum_{\ell=1}^{m}\sum_{\nu=0}^{\ell-1} (-1)^{\nu+\ell}[D^{\ell-\nu-1}eD^\nu(p_\ell LV)]_{x_{j-1}}^{x_j}, \tag{3.6}$$

where the operator L^T is defined by

$$L^Tv = \sum_{\ell=0}^{m} (-1)^\ell D^\ell(p_\ell v) \tag{3.7}$$

If we apply the quadrature rule (2.9) to the left hand side of (3.6), we have

$$\begin{aligned} &\frac{h}{2}\sum_{\ell=1}^{n} \omega_\ell e(\xi_{j\ell})L^TLV(\xi_{j\ell}) \\ &+ \sum_{\ell=0}^{m-1} [\theta_{\ell 1}D^\ell(eL^TLV)(x_{j-1}) + \theta_{\ell 2}D^\ell(eL^TLV)(x_j)](\tfrac{h}{2})^{\ell+1} \\ &= (e,L^TLV) + R_{mn}h^{2r+1}D^{2r}(eL^TLV)(\xi\in I_j), \end{aligned} \tag{3.8}$$

where R_{mn} depends on m and n only.

If we apply the r-point Gauss-Legendre rule to the first term of the richt hand side of (3.6), we obtain

$$\begin{aligned}(3.9)\qquad &\frac{h}{2}\sum_{\ell=1}^{r} \lambda_{\ell r}^{0}\, Le(z_{j\ell})LV(z_{j\ell}) = \\ &= (Le,LV) + S_{mn}h^{2r+1}D^{2r}(LeLV)(\xi\in I_j).\end{aligned}$$

where S_{mn} depends on m and n only. In virtue of (3.4), the left hand side of (3.9) is identically zero. If we combine (3.7) - (3.9) and apply it for the Lagrangian basis functions ϕ_i of $S_0(I_j)$ as defined by (2.15), we get after multuiplication by $2h^{2m-1}$

$$\begin{aligned}(3.10)\qquad &\left|\sum_{\ell=1}^{n} \omega_\ell L^T L\phi_i(\xi_{j\ell})h^{2m}e(\xi_{j\ell})\right| \le C_1 \sum_{\ell=0}^{m-1}(|D^\ell e(x_{j-1})| + |D^\ell e(x_j)|) + \\ &+ h^{2m}\left|\sum_{\ell=0}^{m-1}[\theta_{\ell 1}D^\ell(eL^TL\phi_i)(x_{j-1}) + \theta_{\ell 2}D^\ell(eL^TL\phi_i)(x_j)](\tfrac{h}{2})^\ell\right| + \\ &+ C_2h^{2k+2}[\|eL^TL\phi_i\|_{W^{2r}(I_j)} + \|LeL\phi_i\|_{W^{2r}(I_j)}] \le C(u)h^{k+2}, \\ &\qquad\qquad i = 1,\dots,n.\end{aligned}$$

Analog to (2.22), we can prove that

$$(3.11)\qquad |\omega_\ell L^TL\phi_i(\xi_{j\ell})h^{2m} - 2h^{2m-1}(L\phi_i,L\phi_\ell)| \le Ch^2,$$

which means that, for sufficiently small h, the matrix $(\omega_\ell L^TL\phi_i(\xi_{j\ell}))$ is an $O(h^2)$ perturbation of the positive definite matrix $(2h^{2m-1}(L\phi_i,L\phi_\ell)$, whose eigenvalues and entries are of O(1). This implies that the entries of $(\omega_\ell L^TL\phi_i(\xi_{j\ell})h^{2m})^{-1}$ are of O(1).

<u>THEOREM 3</u>. *Let* $u \in C_0^{m-1}(I) \cap W^{2r}(\Delta)$ *be the solution of* (3.1) *and let* $U \in S_0^{k,m}(\Delta)$ *be the solution of* (3.4). *Then* $e(x) = u(x) - U(x)$ *has the bounds* (3.5) *plus the bounds*

$$\begin{aligned}(3.12)\qquad &|e(\xi_{j\ell})| \le C(u)h^{k+2}, \quad j = 1,\dots,N;\ \ell = 1,\dots,n; \\ &|De(\eta_{j\ell})| \le C(u)h^{k+1}, \quad j = 1,\dots,N;\ \ell = 1,\dots,n+1,\end{aligned}$$

where $\xi_{j\ell}$ *and* $\eta_{j\ell}$ *are given by* (2.19) *and* (2.28), *respectively.*

PROOF. The first part of (3.12) was already established by (3.9) - (3.11). The second part is proved analog to Theorem 2. □

REMARK. RUSSELL and CHRISTIANSEN [8] also gave a proof of (3.12); they proved in another way that the first bound of (3.12) occurs at the interior of the polynomial

$$(3.13) \qquad \int_{-1}^{\sigma} (t-\sigma)^{m-1} P_r^{0,0}(t)\,dt, \qquad \sigma = \frac{2}{h}(x-x_{j-1}) - 1$$

which can be shown to be equal to $(1-\sigma^2)^m P_n^{m,m}(\sigma)$ up to a constant factor. The proof of this equality can be given by using formula (2.6) with $\alpha = \beta = 0$ and elaborating the integral (3.13) which gives the desired result.

4. CONCLUSIONS

In this paper, it was proved for two classes of Galerkin methods that superconvergence also occurs outside the knots of the partition, albeit in a more modest form. Its existence can easily be proved for other classes of problems which are solved by the Ritz-Galerkin or the collocation method. Examples are nonlinear two-point boundary problems and parabolic equations in one space variable [4].

The interior superconvergence is especially important if the finite element space is of degree 2m, because the order of convergence at $\dot{x}_j$ is then the same as at x_j.

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